Practical lattice reductions

for CTF challenges

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Outline

Outline

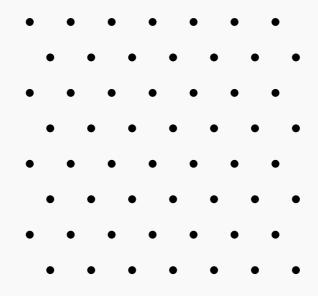
- 1. Lattices?
- 2. Why lattices?
- 3. How lattices?
- 4. Lattice tips and tricks
- 5. Common lattice problems
- 6. Building with lattices
- 7. Get your hands dirty

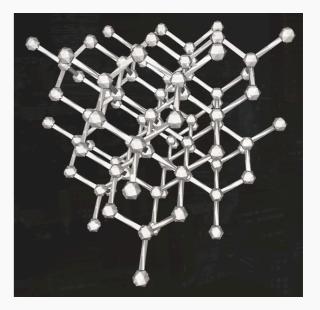
Lattices?

- "A lattice \mathcal{L} of dimension n is a discrete additive subgroup of \mathbb{R}^n ."
- It's a group \Rightarrow addition, scalar mult
- Discrete \Rightarrow we can map it to \mathbb{Q}^n or \mathbb{Z}^n
- A group is a \mathbb{Z} -module, so think vector spaces
- Pick a basis $\pmb{B}=(\pmb{b}_1,...,\pmb{b}_n)\in\mathbb{R}^n$
 - We usually write \boldsymbol{b}_i as rows of \boldsymbol{B}
- $\mathcal{L} = \{\sum a_i \cdot b_i \mid a \in \mathbb{Z}^n\}$
- Many choices of \boldsymbol{B}



https://www.redwoodstore.com/spectrefan-lattice



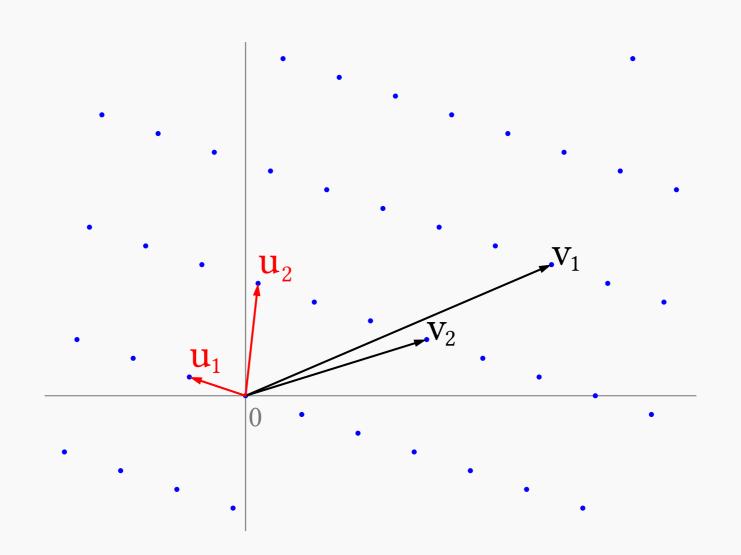


https://en.wikipedia.org/wiki/Lattice_%28 group%29 https://en.wikipedia.org/wiki/File: Diamond_lattice.stl

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Lattices



https://en.wikipedia.org/wiki/Lattice_reduction

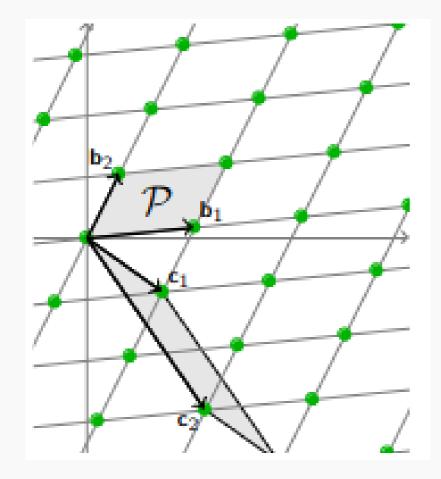
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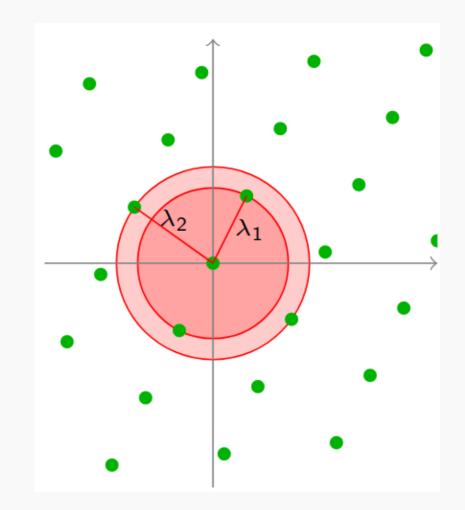
Lattice properties

- "Fundamental parallelepiped" $\mathcal{P}(\boldsymbol{B}):$ a single "enclosed region"
 - $\mathcal{P}(\boldsymbol{B}) = \{\sum a_i \cdot \boldsymbol{b}_i ~|~ \boldsymbol{a} \in [0,1)\}$
 - \mathbb{R}^n is tiled by $\mathcal{P}(\boldsymbol{B})$
- $\det(\mathcal{L}) = \operatorname{vol}(\mathcal{P}(\boldsymbol{B}))) = |\det(\boldsymbol{B})|$
 - $\,\circ\,$ Invariant, independent of ${\pmb B}$
 - Base change: invertible and unimodular
- Successive minima: $\lambda_i(\mathcal{L})$
 - $\lambda_1(\mathcal{L})$ length of shortest vector
 - $\circ \ \lambda_1(\mathcal{L}) \leq \sqrt{n} |\text{det}(\mathcal{L})|^{\frac{1}{n}}$
 - $\circ \ \operatorname{GM}(\lambda_1,...,\lambda_n) = (\prod \lambda_i)^{\frac{1}{n}} \leq \sqrt{n} |\mathrm{det}(\mathcal{L})|^{\frac{1}{n}}$
- Distance $\mu(\boldsymbol{t},\mathcal{L}) = \min_{\boldsymbol{v} \in \mathcal{L}} \lVert \boldsymbol{t} \boldsymbol{v} \rVert$

Lattice properties



https://simons.berkeley.edu/sites/default/files/docs/14953/intro.pdf



https://simons.berkeley.edu/sites/default/files/docs/14953/intro.pdf

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```
from sage.modules.free_module_integer import IntegerLattice
B = Matrix(QQ, [[1, 0], [0, 2]])/2
L = IntegerLattice(B.denominator() * B, lll_reduce=False)
# Warning, may be slow :)
L.shortest_vector() # (1, 0)
L.closest_vector((123/42, 345/12)) # (3, 28)
```

L.volume() # 2

Why lattices?

- Many things in lattices are hard
- SVP: "shortest vector problem"
- CVP: "closest vector problem"
- SIS: "short integer solutions"
- \Rightarrow build trapdoors, e.g. LWE
- Hopefully post-quantum too
- See later

- Many things are "small"
- Many things are discrete
- e.g. some instances of integer programming
- RSA: $pq-\varphi(pq)=p+q-1$ is "small" wrt. pq
- Breaking lattice schemes
- Generally: linear structure
- Think about **linear systems** with small solutions

- Known to break many crypto "weaknesses"
- Some bias in your RNG? Lattices will break it.
- Chose your RSA private key wrong? Lattices will break it.
- Lost some precision in your floating points calculations? Lattices might help.
- Some think they even might break factoring :)

How lattices?

- Starting point: some basis ${\boldsymbol B}$
- Goal: good basis B'
- But what is good?
- And how do we find it?

- Goal: find a better basis
- Good basis?
 - Shorter basis vectors
 - Close to orthogonal
 - Find some short vectors
- Great basis?
 - $\circ \ \operatorname{Read} \operatorname{off} \lambda_1(\mathcal{L})$
 - $\circ \ \operatorname{Read} \ \operatorname{off} \ \operatorname{all} \ \lambda_i(\mathcal{L})?$

- We know $\det(\mathcal{L})$ is constant
- Want: shorter \boldsymbol{b}_i
- So we need wider angles between all \boldsymbol{b}_i to have more area
- Hence, more orthogonal
- Gram-Schmidt orthogonalization
- But breaks the lattice
- Use it as a guideline

Lattice reduction algorithms

- LLL (Lenstra-Lenstra-Lovász)
 - Polynomial time $\mathcal{O}\left(n^6 \log^3 \|\boldsymbol{B}\|_{\infty}\right)$
 - $\circ \|\boldsymbol{b}_1'\| \leq 2^{\frac{n-1}{2}} \lambda_1(\mathcal{L})$
- HKZ (Hermite–Korkine–Zolotarev)
 - Exponential time
 - $\circ \ \| \boldsymbol{b}_1' \| = \lambda_1(\mathcal{L})$
- BKZ (Block (H)KZ)
 - $\,\circ\,$ Parametrized by block size β
 - Larger β : slower
 - Smaller β : worse basis
- Sieving and other costly approaches

- In two dimensions, exact is easy
- Provides some basic intuition for LLL
- Looks like GCD

```
def gauss_reduction(v1, v2):
    while True:
        if v2.norm() < v1.norm():
            v1, v2 = v2, v1 # swap step
        m = round( (v1 * v2) / (v1 * v1) )
        if m == 0:
            return (v1, v2)
        v2 = v2 - m*v1 # reduction step</pre>
```

```
def LLL(B, delta):
   Q = gram schmidt(B)
   def mu(i,j):
       v = B[i]
       u = Q[j]
        return (v*u) / (u*u)
   n, k = B.nrows(), 1
   while k < n:
       # length reduction step
       for j in reversed(range(k)):
            if abs(mu(k,j)) > .5:
                B[k] = B[k] - round(mu(k,j))*B[j]
                Q = gram schmidt(B)
       # swap step
       if Q[k]*Q[k] >= (delta - mu(k,k-1)**2)*(Q[k-1]*Q[k-1]):
            k = k + 1
        else:
            B[k], B[k-1] = B[k-1], B[k]
            Q = gram_schmidt(B)
            k = max(k-1, 1)
   return B
```

Lattice tips and tricks

$$\boldsymbol{z} = a_1 \boldsymbol{v}_1 + \ldots + a_m \boldsymbol{v}_m$$

- m > n
- All a_i small
- Linear system with a small solution
 - But underdetermined
 - So not just linear algebra
- (approximate) SVP would find a short solution

$$egin{pmatrix} m{z} & 0 \ 0 \ ... \ 0 \ -m{v}_1 \ 1 \ 0 \ ... \ 0 \ ... \ 0 \ dots \ \ dots \ \ dots \ \ dots$$

- But, what if we have some remaining short $r = z \sum a_i v_i$?
- To LLL, short is short
- Assign higher weights \boldsymbol{W} to first columns
 - \Rightarrow Wr not so short anymore
 - $\circ~$ Could even vary W_i for element of ${\boldsymbol z}$
- Note: short vectors can also be the negation of what you search

$$\begin{pmatrix} W \boldsymbol{z} & 0 & 0 & \dots & 0 \\ -W \boldsymbol{v}_1 & 1 & 0 & \dots & 0 \\ -W \boldsymbol{v}_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -W \boldsymbol{v}_m & 0 & 0 & \dots & 1 \end{pmatrix}$$

z = vector(ZZ, [...]) v = Matrix(ZZ, m, n, [[...], ...]) L = Matrix.block([[Matrix(z), 0], [v, 1]]) W = Matrix.diagonal([w1, w2, ..., wn, 1, 1, ..., 1]) L = (L * W).LLL() / W

How to determine weights

- 1. Wing it and hope for the best :)
 - Works fairly often
 - Can finetune with synthetic data
 - Wiggling weights can even help finding different solutions
- 2. Investigate expected weights
 - Mostly when different columns have different expected weights in the target vector
 - Observe: outliers in $\|\boldsymbol{v}\| = \sqrt{v_1^2 + \ldots + v_n^2}$ weigh more
 - So the goal is: make all v_i roughly equal
 - Investigate expected result in target vector
 - $\,\circ\,$ Modify weights per column so target vector is all 1 (or arbitrary constant like $2^{128})$

- Sometimes, switch view to CVP
- Rather than solving a linear system, you're close to some lattice point
- e.g. integer multiple + random noise (see LWE later)
- So looking for a lattice vector close to our target
 - Either close vector is final goal
 - Or just solve with linear algebra after
- If you have a range of values, put the target in the center

Kannan embedding

 $\begin{pmatrix} \boldsymbol{B} & \boldsymbol{0} \\ \boldsymbol{t} & \boldsymbol{q} \end{pmatrix}$

- Embed CVP into an SVP instance
- Likely close to what you started with
- Short vector: $(\boldsymbol{t} \boldsymbol{B} \boldsymbol{c}, q)$
- $q \sim$ a weight, matters for results

Babai's closest plane

- Uses a reduced basis for the original lattice
- Greedy algorithm
- Iteratively project each coordinate onto the closest hyperplane
- In sage, over \mathbb{Q} : GS step is generally slow
 - Exact numbers that grow fast-ish

```
def Babai_CVP(mat, target):
    M = IntegerLattice(mat, lll_reduce=True).reduced_basis
    G = M.gram_schmidt()[0]
    diff = target
    for i in reversed(range(G.nrows())):
        diff -= M[i] * ((diff * G[i]) / (G[i] * G[i])).round()
    return target - diff
```

- When LLL is fast enough, but gives no results
- Consider trying BKZ instead
- Experiment with block size $\beta,$ synthetic data is good
- $(\beta = n) \equiv \text{HKZ}$
- For more speed (especially coppersmith): consider flatter

- Got a linear system and some bounds?
- Why not try asking nicely
- rkm0959 made a tool/library: rkm0959/Inequality_Solving_with_CVP
- Could even make a wrapper for convenience
- Good for first exploration, not always foolproof

- Your target is not always the shortest
- Or even in the basis for that matter
- It's still short though
 - It's a small linear combination of basis vectors
 - Try bruteforce
 - Or random combinations
- fp(y)lll also has structured enumeration
- Optionally with extra pruning
- badly documented
 - https://fpylll.readthedocs.io/en/latest/modules.html

```
from fpylll import IntegerMatrix
from fpylll.fplll.gso import MatGS0
from fpylll.fplll.enumeration import (Enumeration,
                                       EvaluatorStrategy)
A = IntegerMatrix.from matrix(M.LLL())
count = 2000
G = MatGSO(A)
G.update gso()
enum = Enumeration(G, nr solutions=count,
          strategy=EvaluatorStrategy.BEST N SOLUTIONS)
n = M.nrows()
for vec, length in enum.enumerate(0, n, max dist, 0, t):
  . . .
```

- Polynomials form a vector space
 - If degree is bounded/fixed
 - Over $\mathbb R$ or otherwise
- So we can take a discrete additive subgroup of them
- Look, ma, it's a lattice!
- Basis for coppersmith's method and ring-LWE (see later)

- Hmm, I have a lattice over \mathbb{F}_2
- While that may be true, "small" mostly breaks down
- Have a look at coding theory instead
- Techniques like *ISD* can be very powerful here
- Can also apply over
 - $\circ \ \mathbb{F}_3$ or other small fields
 - Fields of small characteristic (\mathbb{F}_{2^k} etc)

Common lattice problems

- Instead of working over $\mathbb{Z},$ we now want $\mathbb{Z}/q\mathbb{Z}$
- Keep thinking about linear systems
- $\sum a_i x_i \equiv y \pmod{q}$
- $\sum a_i x_i = y + kq$
- Repeat a few times
- Stack $q \cdot I_m$ under your matrix

• Given:

-

 A set $S = \{s_1,...,s_n\}$
- A value $v = \sum b_i s_i$, with $b_i \in \{0, 1\}$
- Find appropriate b_i
- Often called a knapsack problem
- More accurately it's a subset sum problem
 - there are no values attached
- Known public key cryptosystem
 - Merkle-Damgård
 - Broken by lattices (low density)

Knapsack and Subset Sum

$$\begin{pmatrix} v & 0 & 0 & \dots & 0 \\ -s_1 & 1 & 0 & \dots & 0 \\ -s_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -s_n & 0 & 0 & \dots & 1 \end{pmatrix}$$

- Short vector $(0,b_1,b_2,...,b_n)$
- Rephrase as CVP
 - Leave out first row
 -
 \circ Target: (v,0,0,...,0)
- Optimize CVP:
 - Want $b_i \in \{0, 1\}$, so centered around $\frac{1}{2}$
 - Can do the same trick in the original lattice

Think about it:

- What about a more general version
 - $\circ \ b_i \in \mathcal{X}$
 -
 $\circ\,$ Knapsack: optimize for some value
 t_i
 - Dealing with negative numbers
 - Parallel instances
 - Modular

•

- *Hidden* Subset Sum problem
- Don't forget to look at the negatives in your reduced basis!

- Given: samples $x_i = q_i p + r_i$, with small r_i
- Target: Find p, the gcd of the samples, up to errors r_i
- Partial AGCD: $r_0 = 0$
 - i.e. $x_0 = q_0 p$
 - e.g. RSA with extra information

- $\frac{x_i}{x_0} \approx \frac{q_i}{q_0}$
- Find candidate q_0
- Recover p from $x_0, q + 0$
- Short vector: $(Wq_0,q_0r_1-q_1r_0,\ldots)$

$$\begin{pmatrix} W & x_1 & x_2 & \dots & x_n \\ 0 & x_0 & 0 & \dots & 0 \\ 0 & 0 & x_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & x_0 \end{pmatrix}$$

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• Orthogonal lattice

- $\circ \mathcal{L}^{\perp} = \{ \boldsymbol{v} \mid \forall b \in \mathcal{L}, \langle \boldsymbol{v}, \boldsymbol{b} \rangle = 0 \}$
- Observe: $\mathcal{L} \subseteq (\mathcal{L}^{\perp})^{\perp}$
- Useful for some problems, including hidden subset sum
- General idea: find short vector *orthogonal* to some target
 - Often just to one vector or a lattice with 2 basis vectors
 - Then derive some useful quantity
- $\mathcal{L}(\boldsymbol{q},\boldsymbol{r})^{\perp} \subseteq \mathcal{L}(\boldsymbol{x})^{\perp}$, and short
- reduce $\mathcal{L}({m x})^\perp$ to find a sub-basis spanning $\mathcal{L}({m q},{m r})$
- recover q, r

$$\alpha_i x + \rho_i k_i \equiv \beta_i \pmod{N}$$

- k_i bounded
- See (EC)DSA biased nonce attacks
 - $\circ \ \alpha_i = -r_i, \rho_i = s_i, \beta_i = H 2^t \text{MSB}_{\text{nonce}}, k_i = \text{LSB}_{\text{nonce}}$
- Key realization: $k_i \equiv \frac{\beta_i \alpha_i x}{\rho_i}$ is bounded/small
 - Try to build the lattice ;)
 - Or read biased nonce papers
- Generalization: EHNP
 - Support multiple "holes"
 - Formulation gets complex
 - "Just" implementing the paper is feasible

- Shift in focus
 - No longer linear systems... one polynomial
- $f(x) \equiv 0 \pmod{N}$, monic, x < X bounded
 - $\circ~$ Or even mod $d\approx N^\beta$ with $d\mid N$
- Sage has f.small_roots(), with some parameters
 - or use implementation from kiona/defund/...
 - flatter usually works very well for these
- $X < N^{rac{eta^2}{\deg(f)} arepsilon}$
 - $\circ~\varepsilon$ is a useful parameter for sage
 - $\,\circ\,$ Smaller ε is slower, may be brute some bits
- Multivariate (heuristic) generalizations

Coppersmith intuition

- Generate polynomials sharing roots $(\bmod \, N)$
 - $\circ \ x^k f(x)$
 - $\circ N^k f(x)$
 - $\circ f^k(x)$
 - $\circ \ \Rightarrow x^i N^j f^k(x)$
- Find small f' over $\mathbb Z$
 - Lattice reduction
 - Factor over \mathbb{Z}
 - $\circ \ \ {\rm Check \ results} \ ({\rm mod} \ N)$
- Multivariate
 - Extract roots from multiple polynomials
 - Gröbner basis, resultants, ...

• RSA: stereotyped message

 $\circ \ f(x) = \left(K + x\right)^e - c \equiv 0 \pmod{N}, x \text{ small}$

• RSA: partially known factor

$$\circ \ f(x) = \left(p_{\mathrm{high}} + x\right) \equiv 0 \pmod{p}, x < N^{\frac{1}{4}}, p \mid N$$

• Boneh-Durfee

$$\label{eq:generalized_states} \begin{array}{l} \circ \ f(x,y) = x((N+1)-y) \equiv 0 \pmod{e} \\ \circ \ y = -(p+q), x \mbox{ modular "wraps"} \end{array}$$

- AGCD
 - Multivariate

Building with lattices

Learning With Errors

$$s \leftarrow \chi_{k}^{n}$$
$$a_{i} \leftarrow \mathbb{Z}_{q}^{n}$$
$$e_{i} \leftarrow \chi_{e}$$
$$b_{i} = \langle s, a_{i} \rangle + e_{i}$$

$$b = As + e$$

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- Distinguishing
- Key recovery
- Embedding messages

•
$$b = \langle \boldsymbol{s}, \boldsymbol{a} \rangle + e + \frac{q}{p} \cdot m$$

• $b = \langle \boldsymbol{s}, \boldsymbol{a} \rangle + p \cdot e + m$

$$\begin{split} R &= \mathbb{Z}[X]/f(X), f \text{ monic irreducible, } \deg(f) = N \\ R_q &= R/qR \\ s(X) \leftarrow \chi_k^N \in R_q \\ e(X) \leftarrow \chi_e^N \in R_q \\ b_i(X) &= a_i(X) \cdot s(X) + e_i(X) \end{split}$$

$$b_i(X) = a_i(X) \cdot s(X) + e_i(X)$$

$$\boldsymbol{b_i} = \begin{pmatrix} a_{i,1} & (X^{-1}a)_1 & \dots & (X^{-N+1}a)_1 \\ a_{i,2} & (X^{-1}a)_2 & \dots & (X^{-N+1}a)_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{i,N} & (X^{-1}a)_N & \dots & (X^{-N+1}a)_N \end{pmatrix} \cdot \boldsymbol{s} + \boldsymbol{e_i}$$

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$$\begin{split} R &= \mathbb{Z}[X]/(X^N \pm 1) \\ R_q &= R/qR \\ f \leftarrow \chi^N \in R_q, \exists f^{-1} \\ g \leftarrow \chi^N \in R_q \\ h &= \frac{g}{f} \end{split}$$

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- Distinguishing
- Recover f
- Embedding messages

$$\begin{array}{l} \circ \quad f = p \cdot f' + 1 \\ \circ \quad c = \frac{g}{f} + \frac{q}{p} \cdot m \\ \circ \quad c \cdot f \equiv g + \frac{q}{p} \cdot m \cdot p \cdot f' + \frac{q}{p} \cdot m \end{array}$$

• Alternative:

•
$$h = p \cdot \frac{g}{f}, c = r \cdot h + m$$

• Parameters matter: NTRU fatigue/overstretched NTRU

$$\begin{pmatrix} 1 & h \\ 0 & q \end{pmatrix}$$

- Matrix form for h (and 0, 1, q)
 - (anti-)circulant matrix
- short vector: $(f,g)=f\cdot(1,h)+k\cdot(0,q)$

- https://github.com/malb/lattice-estimator
 - Viability
 - Ideas for attacks to look at
 - Making sure your own lattices are safe?
- https://github.com/WvanWoerden/NTRUFatigue
 - Specifically for NTRU
 - Fatigue/overstretched regime

Linear algebra

- Basis is linear algebra + noise
- Sometimes the noise is not there
- Or not enough
- So just throw matrices at it

Lattice reduction

- AKA primal attack
- The straightforward thing

Weak structure

- RLWE/NTRU/... in a weird ring
 - Composite modulus
 - Reducible polynomial

••••

- Chinese remainder theorem
 - AKA working mod a factor
- Depends on end goal

Linearization

- Arora-Ge attack
- Consider $e \in \{-1, 0, 1\}$
- Write $v = \langle s, a \rangle b$
- $\bullet \ v \cdot (v-1) \cdot (v+1) \equiv 0$
- Max degree: 3
- Enough samples \rightarrow each monomial becomes 1 linear variable
- Linear algebra

Some resources

- https://eprint.iacr.org/2023/032.pdf
- https://eprint.iacr.org/2020/1506.pdf
- https://kel.bz/post/lll/
- https://github.com/rkm0959/Inequality_Solving_with_CVP
- https://github.com/jvdsn/crypto-attacks
- https://github.com/kionactf/coppersmith
- $\bullet\ https://gist.github.com/RobinJadoul/796857fa33b118c17a4e54ff1b7ccfbe$
- https://doi.org/10.1007/3-540-44670-2_12

Get your hands dirty

ImaginaryCTF (round 25)

A multiplicative knapsack, kinda.

Author Robin_Jadoul
Flag format ictf{...}

ImaginaryCTF 2023

tan(x) is a broken hash function in terms of collision resistance and second
preimage resistance. But you surely can't find the preimage of tan(flag),
right?

Author maple3142
Flag format ictf{...}

ImaginaryCTF 2023

I threw in a bit of source-given rev, because why not.

> *I* hate crypto and rev because both are math

Sorry to people who feel like this and even say so in the ictf discord ;)

Author Robin_Jadoul
Flag format ictf{...}

ECSC 2023

Champagne for my real friends, real pain for my sham friends.

Author Robin_Jadoul Flag format ECSC{...}

ICC 2022

I want to keep my private key small, but I've heard this is dangerous. I think I've found a way around this though!

Author jack Flag format ICC{...}

pbctf 2020

I know there's a famous attack on biased nonces. Then, how about this?

Author rbtree
Flag format pbctf{...}
Extra note (try the harder approach, just for fun)

pbctf 2021

I came up with this fun game that only lucky people can win. Do you feel lucky?

Author UnblvR
Flag format pbctf{...}

SECCON finals 2022

Recently, I learned that this random number generator is called "MRG".

Author Xornet Flag format SECCON{...}

SEETF 2023

How to bypass this line?

assert __import__('re').fullmatch(r'SEE{\w{23}}', flag:=input()) and not int.from_bytes(flag.encode(), 'big') % 13**37

Author Neobeo Flag format SEE{...}

TSJ CTF 2022

I encrypted the flag and messages by xoring them with a random number generator again. But it should be harder to break this time.

Author maple3142 Flag format TSJ{...}

HITCON CTF 2023

I think you already know what is this challenge about after seeing the challenge name :)

Author maple3142 Flag format hitcon{...}

<Mostly new>

Based upon the challenge reality from google ctf 2019

Author Robin_Jadoul
Flag format flag{...}